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Field-induced circulation flow in magnetic fluids

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In this paper, we present results of a theoretical study of circulation flow in ferrofluids under the action of an alternating inhomogeneous magnetic field. The results show that the field with the amplitude of about $17\,\mathrm{kA\,m^{-1}}$ and angular frequency $10\,\mathrm{s^{-1}}$ can induce mesoscopic flow with a velocity amplitude of about $0.5\,\mathrm{mm\,s^{-1}}$. This mechanism can be used for intensification of drag delivery in blood vessels.

This article is part of the theme issue 'Patterns in soft and biological matters'.

1. Introduction

The main problem of treatment of brain strokes is related to very slow diffusion of the thrombolytic drugs toward blood clots through blocked vessels. An American company Pulse Therapeutics has proposed a smart solution to this problem using magnetic micro- or nanoparticles entrained in motion by alternating magnetic fields and able to create recirculating flows in the blocked vessels [1]. These recirculating flows considerably enhance convective transport of the drug towards the clots. Only a few works have been published on this topic [2–4], so, the physical understanding of the origin of the recirculating or oscillatory flows created by moving and rotating magnetic particles is still lacking. To shed more light onto this problem, in

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this paper, we propose a theoretical model considering ferroparticle motion and induced flows inside a channel under applied alternative non-homogeneous magnetic fields. The obtained amplitudes of the velocity and fluid displacement are compared with those required for efficient drug delivery to the blood clots in a real situation.

The aim of this work is theoretical study of circulation flows induced by a rotating magnetic field in a magnetic fluid drop injected into a liquid filling a flat gap, whose size in its plane is much more than the distance between the gap walls.

2. Mathematical model and the main approximations

We consider an infinite flat gap of the thickness l filled with non-magnetic liquid, containing a drop of a ferrofluid consisting of identical spherical ferroparticles. The rotating magnetic field is created by four solenoids, situated as it is illustrated in figure 1.

Let m and M be the magnetic moment and saturated magnetization of the particle, respectively. It is supposed that the gap thickness l is much less than the solenoid diameter D, as well as the distances a,b shown in figure 1 ($l \ll a, b, D$).

We will denote the local volume concentration of the particles as (x, z, t) and, for maximal simplification of mathematics, consider the two-dimensional approximation, when all physical events take place in the plane (x,z) demonstrated in figure 1. Additionally, we will neglect effects of the particles' Brownian rotation. This means, we suppose that the Zeemen energy of the particle interaction with the field $\mathbf{H}$ is significantly more than the thermal energy kT. Obviously, from the practical point of view, this case is the most interesting. Note, for the magnetite particles with diameter of about 10–20 nm this condition is fulfilled if the local H exceeds $10\,\mathrm{kA\,m^{-1}}$. That is an easily achievable range of the field.

Equations of the magnetizable fluid flow at the low Reynolds number can be presented as [5,6]

$$\left.\begin{aligned}
\rho\frac{\partial v_x}{\partial t} &= -\frac{\partial p}{\partial x} + \eta\Delta v_x + \frac{1}{2}\frac{\partial}{\partial z}\Gamma + \mu_0 M\Phi\left[\cos\theta\frac{\partial}{\partial z} + \sin\theta\frac{\partial}{\partial x}\right]H_x \\
\rho\frac{\partial v_z}{\partial t} &= -\frac{\partial p}{\partial z} + \eta\Delta v_z - \frac{1}{2}\frac{\partial}{\partial x}\Gamma + \mu_0 M\Phi\left[\cos\theta\frac{\partial}{\partial z} + \sin\theta\frac{\partial}{\partial x}\right]H_z \\
&\frac{\partial}{\partial x}v_x + \frac{\partial}{\partial z}v_z = 0
\end{aligned}\right\} \tag{2.1}$$

Here, $\Delta = \partial^2/\partial x^2 + \partial^2/\partial z^2$ is the Laplace operator; $\Gamma = \mu_0 M\Phi(H_z\sin\theta - H_x\cos\theta)$ is the magnetic torque, acting by unit volume of the ferrofluid; $\mathbf{H}$ is the local magnetic field in the fluid; θ is the angle between the particle magnetic moment m and the axis Oz, normal to the gap plane (figure 1). The third terms in the first two equations of (2.1) present the stress, which appear because of the magnetic torque Γ; the fourth ones are the components of the ponderomotive force, acting on the fluid from the side of the non-uniform field $\mathbf{H}$.

The boundary conditions for (2.1) are

$$\left.\begin{aligned}
v_x = v_z = 0 \quad &\text{at } z = 0, l \\
v_x, v_z \to 0 \quad &\text{at } x \to \infty.
\end{aligned}\right\} \tag{2.2}$$

In the 'no Brownian' approximation, the equation for the angle θ reads [5]

$$\frac{\partial\theta}{\partial t} = \frac{1}{2}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) - \frac{1}{6\eta\Phi}\Gamma. \tag{2.3}$$

Note, the ratio Γ/Φ is magnetic torque, which acts on the united volume of the particle.

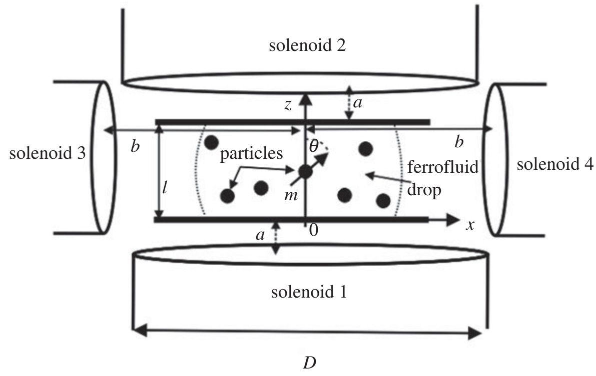

Figure 1. Illustration of the model system.

Equation of continuity for the particles concentration can be written as

$$\left.\begin{aligned}&\frac{\partial}{\partial t}\Phi+\operatorname{div}\left[\Phi\left(\mathbf{v}-\frac{1}{3\pi\eta d}\nabla U\right)\right]=K\Delta\Phi\\&U=-\mu_0 M V_p(H_z\cos\theta+H_x\sin\theta).\end{aligned}\right\}\tag{2.4}$$

Here d is the particle diameter; K is diffusion coefficient; U is the particle potential energy in the field $\mathbf{H}$.

Below we will use the following estimates for the system parameters. We suppose that the particles are magnetite, therefore $M \approx 500\,\mathrm{kA\,m^{-1}}$; the diameter of the particle $d \approx 15$–$20\,\mathrm{nm}$; the volume concentration of the particles $\Phi \sim 0.01$–0.05; the angular frequency of the alternating field, created by the solenoid, ω, is approximately $10\,\mathrm{s^{-1}}$; the solenoid provides the field $\mathbf{H}$ inside the gap with the strength absolute value $H > 10\,\mathrm{kA\,m^{-1}}$. The viscosity and density of the host medium are close to those of water ones, i.e. we suppose $\eta \sim 10^{-3}\,\mathrm{Pa\cdot s}$; $\rho \sim 10^{3}\,\mathrm{kg\,m^{-3}}$. The gap thickness is estimated as $l \sim 1\,\mathrm{mm}$.

Analytical solution of the whole system (1–4) is impossible. However, serious simplifications can be achieved by using the mentioned estimates for the system characteristics. First of all, from the mathematical point of view, the fluid flow is provoked by the last two terms in the Navier–Stokes equations of (2.1). These terms are proportional to the small concentrations Φ of the particles. Therefore, roughly, the scaling relation $v \sim \Phi$ must be held and the terms $1/2(\partial v_x/\partial z - \partial v_z/\partial x)$ and Φv can be considered as the smallest ones in equations (2.3) and (2.4). We will neglect these terms.

Let us present the local concentration of the particles as $\Phi(x,z,t) = \Phi_0(x,z) + \varphi(x,z,t)$ where $\Phi_0 \sim 0.01$–0.05 is the initial volume concentration of the particles in the ferrofluid drop. Estimates show that in the field with the strength $H \sim 10\,\mathrm{kA\,m^{-1}}$ and the spatial scale $D \sim 10^{-2}\,\mathrm{m}$, the characteristic time $\tau = 3\pi\eta d K^2/\mu_0 M V_p H$ of magnetophoretic migration of the magnetite particle with diameter 20 nm at the distance D is more than $5\cdot 10^5\,\mathrm{s} \approx 140\,\mathrm{h}$. That significantly exceeds the time, presenting interest from the viewpoint of the drug delivery technology. Therefore, at least in the first approximation, one can neglect the term φ as compared with Φ_0 and suppose that for the time, presenting interest, the equality $\Phi = \Phi_0(x,z)$ is held. In principle, the drop shape can be varied because of the convective motion of the ferrofluid particles in the generated circulation flow. Discussion of this factor is provided in the Conclusions.

Neglecting the term $1/2(\partial v_x/\partial z - \partial v_z/\partial x)$ and by using the explicit form for the torque Γ, one can rewrite equation (2.3) as

$$\frac{\partial\theta}{\partial t} = -\frac{\mu_0 M}{6\eta}(H_z\sin\theta - H_x\cos\theta).\tag{2.5}$$

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For convenience, one can present the first two equations of (2.1) as

$$\left.\begin{aligned}\rho\frac{\partial v_x}{\partial t} &= -\frac{\partial p}{\partial x} + \eta\Delta v_x + \mu_0 M\Phi G_x,\\ \rho\frac{\partial v_z}{\partial t} &= -\frac{\partial p}{\partial z} + \eta\Delta v_z + \mu_0 M\Phi G_z,\\ G_x &= \frac{1}{2}\frac{\partial}{\partial z}(H_z\sin\theta - H_x\cos\theta) + \left[\cos\theta\frac{\partial}{\partial z} + \sin\theta\frac{\partial}{\partial x}\right]H_x\\ G_z &= -\frac{1}{2}\frac{\partial}{\partial x}(H_z\sin\theta - H_x\cos\theta) + \left[\cos\theta\frac{\partial}{\partial z} + \sin\theta\frac{\partial}{\partial x}\right]H_z.\end{aligned}\right\}\tag{2.6}$$

We will suppose that all solenoids create the field with the same angular frequency; the total field components are

$$\left.\begin{aligned}H_x &= (H_{01x}(x,z) + H_{04x}(x,z))\cos\omega t + (H_{02x}(x,z) + H_{03x}(x,z))\sin\omega t\\ H_z &= (H_{01z}(x,z) - H_{04z}(x,z))\cos\omega t + (-H_{02z}(x,z) + H_{03z}(x,z))\sin\omega t.\end{aligned}\right\}\tag{2.7}$$

Here, $\boldsymbol{H}_{01}, \boldsymbol{H}_{02}, \boldsymbol{H}_{03}$ and $\boldsymbol{H}_{04}$ are amplitudes of the fields created by solenoids respectively with the numbers 1–4 in figure 1. The relations (2.7) mean that the north pole of the solenoid 1 in figure 1 is situated in front of the south pole of the solenoid 2. The magnets with numbers 3 and 4 are turned to each other by the same poles.

It follows from equation (2.5) that, in the order of magnitude, the characteristic time τ_θ of the angle θ relaxation to the given field H is $\tau_\theta \sim \eta/\mu_0 MH$. Supposing that absolute value of the field in the gap has the typical magnitude $H \sim 10\,\mathrm{kA\,m^{-1}}$, the particles are magnetite ($M \sim 500\,\mathrm{kA\,m^{-1}}$) and the carrier liquid is water ($\eta \sim 10^{-3}\mathrm{Pa\cdot s}$), one gets $\tau_\theta \sim 10^{-6}\,\mathrm{s}$. The typical value of the angular frequency for the experiments like that is $\omega \sim 10\,\mathrm{s}^{-1}$, therefore, the field alternation period is about $0.6\,\mathrm{s}^{-1}$. Therefore, the rate of the field alternation is about five-six decimal orders less than the rate of the angle θ relaxation. This means that in any moment of time the angle θ has practically the value θ^0, equilibrium with respect to the field $\mathbf{H}$ at the given moment of time t. This angle is determined by the relations

$$\left.\begin{aligned}&\cos\theta^0 = \frac{H_z}{H}, \sin\theta^0 = \frac{H_x}{H}\\ &H = \sqrt{H_x^2 + H_z^2}.\end{aligned}\right\}\tag{2.8}$$

Taking that into account, we use $\theta = \theta^0$ in equation (2.6). In this approximation, taking into account the standard magnetostatic relation $\mathrm{rot}\boldsymbol{H} = 0$, one gets from (2.6)

$$G_x = \frac{\partial}{\partial x}H; G_z = \frac{\partial}{\partial z}H.\tag{2.9}$$

This is convenient to introduce the current function Ψ so that

$$v_x = \frac{\partial}{\partial z}\Psi, \quad v_z = -\frac{\partial}{\partial x}\Psi.\tag{2.10}$$

Combining (2.6) and (2.10), and taking into account equation (2.9), then after transformations we come to the following equation with respect to Ψ:

$$\left.\begin{aligned}&\rho\frac{\partial Q}{\partial t} = \eta\Delta Q + \mu_0 M\left[\frac{\partial}{\partial x}H\cdot\frac{\partial}{\partial z}\Phi - \frac{\partial}{\partial z}H\cdot\frac{\partial}{\partial x}\Phi\right]\\ &Q = \Delta\Psi.\end{aligned}\right\}\tag{2.11}$$

The boundary conditions for equation (2.11) are

$$\left.\begin{aligned}&z = 0, l \quad \frac{\partial\Psi}{\partial x} = \frac{\partial\Psi}{\partial z} = 0\\ &t = 0, \quad \Psi = 0.\end{aligned}\right\}\tag{2.12}$$

The spatial scale of the velocity v in the x-direction is determined by the parameters a, b and D; the scale in the z-direction is the gap thickness l. Taking into account the strong inequalities $a, b, D \gg l$, we come to the following relation:

$$\frac{\partial^2 \Psi}{\partial z^2} \gg \frac{\partial^2 \Psi}{\partial x^2}$$

Therefore, in equation (2.8) one can put

$$Q = \frac{\partial^2 \Psi}{\partial z^2}; \quad \Delta Q = \frac{\partial^4 \Psi}{\partial z^4}, \tag{2.13}$$

and to present this equation as

$$\left.\begin{aligned} \frac{\partial Q}{\partial t} &= \nu \frac{\partial^2 Q}{\partial z^2} + f; \\ f &= \frac{\mu_0 M}{\rho}\left[\frac{\partial}{\partial x} H \cdot \frac{\partial}{\partial z}\Phi - \frac{\partial}{\partial z} H \cdot \frac{\partial}{\partial x}\Phi.\right] \end{aligned}\right\} \tag{2.14}$$

3. Solution of equation (2.11); results and discussion

Because of the strong inequalities a, b, $D \gg l$, the spatial scale of the function f in the x-direction is much more than that in the z-direction. Thus, at the distance l, the function f changes very weakly and, in the first approximation, inside the gap one can neglect the dependence of f on the coordinate z.

Note that the function f is periodical with the period $2\pi/\omega$. Because of the linearity of equation (2.14), the function Ψ also must be periodical with the same period. Taking this into account, we can expand both f and Ψ in the Fourier series as (for example, ref. [7]):

$$\left.\begin{aligned} f &= \frac{1}{2} fC_0 + \sum_{n=1}^{\infty} (fC_n \cos(\omega n t) + fS_n \sin(\omega n t)) \\ fC_n &= \frac{1}{T}\int_{-T/2}^{T/2} f(x,t)\cos(\omega n t)\,\mathrm{d}t; \quad fS_n = \frac{1}{T}\int_{-T/2}^{T/2} f(x,t)\sin(\omega n t)\,\mathrm{d}t \\ \Psi &= \frac{1}{2}\Psi C_0 + \sum_{n=1}^{\infty}(\Psi C_n \cos(\omega n t) + \Psi S_n \sin(\omega n t)). \end{aligned}\right\} \tag{3.1}$$

Combining equations (3.1), (2.13) and (2.14), we come to the relations:

$$\left.\begin{aligned} \Psi C_n &= -A_n \frac{\mathrm{e}^{kz}\cos kz}{2k^2} - B_n \frac{\mathrm{e}^{kz}\sin kz}{2k^2} - C_n \frac{\mathrm{e}^{-kz}\cos kz}{2k^2} - D_n \frac{\mathrm{e}^{-kz}\sin kz}{2k^2} + L1_n z + L2_n - \frac{fS_n}{2(\omega n)} z^2 \\ \Psi S_n &= A_n \frac{\mathrm{e}^{kz}\sin kz}{2k^2} - B_n \frac{\mathrm{e}^{kz}\cos kz}{2k^2} - C_n \frac{\mathrm{e}^{-kz}\sin kz}{2k^2} + D_n \frac{\mathrm{e}^{-kz}\cos kz}{2k^2} + N1_n z + N2_n + \frac{fC_n}{2(\omega n)} z^2 \\ k &= \sqrt{\frac{\omega}{2\nu}}. \end{aligned}\right\} \tag{3.2}$$

Here $A_n, \ldots, N2_n$ are constants of integration, to be determined.

The boundary conditions (2.12) can be rewritten in the form

$$z = 0, l \quad \frac{\partial \Psi C_n}{\partial z} = 0, \frac{\partial \Psi S_n}{\partial z} = 0, \Psi C_n = 0, \Psi S_n = 0. \tag{3.3}$$

Combining equations (3.2) and (3.3), we come to a system of eight linear equations with respect to the eight variations $A_n, \ldots, N2_n$. Analytical solution of this system is too cumbersome; however, the numerical calculations do not present any problem.

Having the coefficients $A_n, \ldots, N2_n$ found by using equation (2.10), we determine the velocity components $v_{x,z}$.

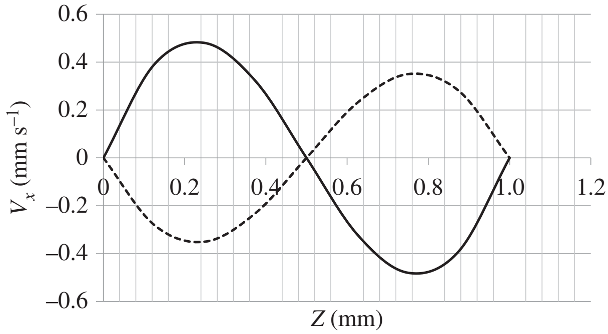

Figure 2. Longitudinal velocity component v_x versus z-coordinate at a fixed $x = 10$ mm in figure 1. Dashed line: at time $t = 0.5$ s; solid line: $t = 1$ s. The field angular frequency $\omega = 10$ s^{-1}. Volume concentration of the particles in the drop centre $\Phi_0 = 0.01$; the dispersion $\sigma = 1$ cm; the solenoid characteristics: diameter $D = 1$ cm; current $I = 10$ A; height $h = 1$ cm; number of coils $N = 10^4$ for all solenoids. The distances, shown in figure 1: $a = b = 5$ cm; the gap thickness $l = 1$ mm.

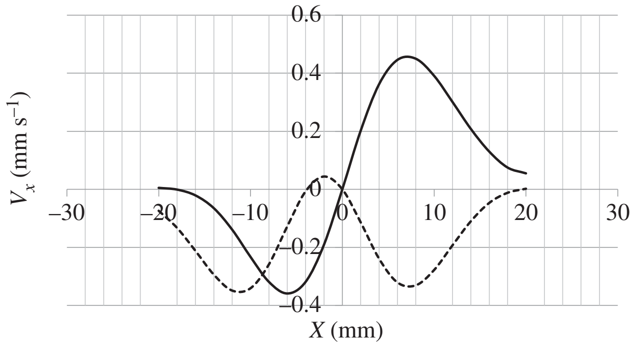

Figure 3. Longitudinal velocity component v_x versus x-coordinate at a fixed $z = l/8$ in figure 1. Dashed line: at time $t = 0.5$ s; solid line: $t = 1$ s. The other parameters are the same as in figure 2. Note that for the system parameters, corresponding to figures 2 and 3, the amplitude of absolute value of the field in the gap centre is about 17 kA m^{-1}.

For definiteness, we suppose that the spatial distribution of the particles obey the Gaussian distribution

$$\Phi_0(x, z) = \Phi^0 \exp\left[-\frac{(x^2 + (z - l/2)^2)}{\sigma^2}\right]. \tag{3.4}$$

This formula means that in the vicinity of the middle of the gap there is a dense cloud of the ferrofluid with the maximal volume concentration Φ^0 at the point $(0, l/2)$; σ is dispersion of this distribution, i.e. characteristic size of the drop. It should be noted that the spatial distribution (3.4) is used here just as a model one. Any other, more relevant to a concrete experimental situation, can be used for the calculations in the frame of the suggested approach.

To calculate the components of amplitude $\boldsymbol{H}_{01}$ of the field, created by the solenoid 1 in figure 1, we use the Biot–Savart Law. In cylindrical coordinates, it has the form

$$H_{01x}(x, z) = \frac{I \cdot D \cdot N}{8\pi h} \int_{-(h+a)}^{-a} \mathrm{d}z' \left(\int_0^{2\pi} \frac{(z - z')\cos\varphi}{[(z - z')^2 + ((D/2)\sin\varphi)^2 + (x - (D/2)\cos\varphi)^2]^{3/2}} \mathrm{d}\varphi \right)$$

$$H_{01z}(x, z) = \frac{I \cdot D \cdot N}{8\pi h} \int_{-(h+a)}^{-a} \mathrm{d}z' \left(\int_0^{2\pi} \frac{(D/2) - x\cos\varphi}{[(z - z')^2 + ((D/2)\sin\varphi)^2 + (x - (D/2)\cos\varphi)^2]^{3/2}} \mathrm{d}\varphi \right)$$

where I is the current in the solenoid, h is its height, N is the number the solenoid coils and a is the distance from solenoid to the gap. This formula represents the field components for the lower solenoid, for the other solenoids these components have been determined similarly.

Some results of calculations of the longitudinal component v_x of the fluid velocity are shown in figures 2 and 3.

4. Conclusions

We present a theoretical model of magnetically induced circulation flow in a flat channel filled by a Newtonian liquid and a drop of a ferrofluid. The results show that a rotating magnetic field with amplitude about $17\,\mathrm{kA\,m^{-1}}$ and angular frequency of about $10\,\mathrm{s^{-1}}$ in the channel with the wideness 1 mm can induce circulating flow with the velocity amplitude of about $0.5\,\mathrm{mm\,s^{-1}}$.

Let us discuss now the approximation of the permanent shape of the ferrofluid drop. Since the magnetophoretic displacement of the ferrofluid particles is neglected, the drop shape can be changed only because of the convective effects. In the order of magnitude, the displacement δr of the drop particles for the field period satisfies the relation $\delta_r \leq v_{\max} 2\pi/\omega$, where $v_{\max}$ is the maximal value of the velocity. Our results give $v_{\max} \sim 0.5\,\mathrm{mm\,s^{-1}}$. By using $\omega \sim 10\,\mathrm{s^{-1}}$, one gets $\delta_r \leq 0.3$ mm. This is much less than the used characteristic size $\sigma \sim 1$ cm of the drop. Therefore, the simplification of the permanent shape of the drop is justified, at least as the first approximation.

Note that in the frame of the presented model the 'horizontal' solenoids 3 and 4 in figure 1 are quite enough to induce the circulating flow. However, in practice, dealing with the real blood vessel, it is very difficult to provide the solenoids dispositions exactly along the vessel axis. That is why we have considered the configuration of the solenoids, which is more realistic from the viewpoint of the medical application.

Data accessibility. This article has no additional data.

Authors' contributions. P.K. and G.V.-D.: the physical idea of the study. A.Z.: mathematical model. A.M. and M.R.-M.: calculations.

Competing interests. We declare we have no competing interests.

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